Practical theory to compute the microwave instability threshold

Ryan R. Lindberg

Advanced Photon Source, Argonne National Laboratory,
Argonne, IL 60439, USA
E-mail: lindberg@aps.anl.gov

We present a rather accurate method to compute the microwave instability threshold that is computationally relatively simple to perform. We derive the integral equation upon which the method is based, and simplify it by approximating the longitudinal motion by that of a simple harmonic oscillator. The stable frequencies can be found as a function of current using standard eigenvalue solvers, while we identify the microwave instability threshold as the lowest current at which a pair of the stable frequencies cease to exist. Our method appears to represent a computationally fast and accurate way to compute the bunched-beam microwave instability threshold in storage rings, and we show examples for both the shielded and unshielded coherent synchrotron impedance.

Keywords: Microwave instability; Coherent synchrotron radiation.

1. Introduction

Study of the microwave instability for bunched beams has a long history [1]. One important development was Boussard's observation [2] that because the microwave instability develops over small scales, one might usefully apply the coasting beam theory of Refs. [3,4] by replacing the average current with the peak current. The resulting criterion is sometimes called the Boussard-Keil-Schnell condition, and while easy to compute it typically underestimates the microwave instability threshold current by a factor of 2 or more. At about this same time Sacherer developed a matrix analysis of the Vlasov equation that identified the microwave instability as resulting from classical mode-coupling [5,6]. Since that time a number of related approaches have been put forward, including more formalized Boussard-type analyses in [7,8], along with matrix-type calculations in [9–12]. The former tend to offer quick estimates, while the latter may require significant computational effort. This paper presents a method for obtaining a rather accurate estimate for the microwave instability threshold current using a modest level of computation.

Our calculation was partly inspired by the work in Refs. [7,8,13], and we also derive a difficult-to-solve integral equation governing stability. Rather than turning to approximations, we show how the current for a given stable frequency can be determined by a relatively easy to solve eigenvalue problem. In this way we can determine the current at which two initially real and distinct frequencies merge and cease to exist, which we associate with longitudinal mode-coupling [5,6] and the onset of the microwave instability. The majority of this paper is spent deriving the integral equation that governs longitudinal stability and describing how we propose to solve it, but we also apply it to predict the microwave instability threshold for both an unshielded and shielded CSR impedance. Future work will apply the theory to other impedances.

2. Theory

We will use the longitudinal wakefield $W_{\|}(z)$ to quantify the one-turn energy loss of a test particle located at z due to the fields generated by a drive particle at $z = 0$. Here, $z = s - ct$ is the co-moving longitudinal coordinate, where s is the independent variable along the ring, t is the arrival time of the particle, and c is the speed of light. For the usual wakefields that result from the interaction of an ultra-relativistic particle's Coulomb field with changes in the vacuum chamber geometry and/or its finite resistivity, $W_{\|}(z > 0) = 0$ due to causality; wakefields that describe space charge or radiation effects such as coherent synchrotron radiation (CSR) may be non-vanishing when $z > 0$. The Fourier transform of $W_{\|}(z)$ with respect to time is the longitudinal impedance, so that

$$W_{\|}(z) = \frac{c}{2\pi}\int dk\, e^{ikz} Z_{\|}(k). \tag{1}$$

We will investigate the onset of the microwave instability in the presence of $Z_{\|}(k)$ using the linearized Vlasov equation. Hence, we will consider the evolution of the one-particle longitudinal distribution function $F(z, p_z; s)$, where the Hamiltonian longitudinal coordinates are taken to be the position $z = s - ct$ and the momentum $p_z = -(\gamma - \gamma_0)/\gamma_0$, with γ and γ_0 being the Lorentz factors of the particle and the reference orbit, respectively; we choose the negative energy deviation rather than the usual $\delta = (\gamma - \gamma_0)/\gamma_0$ so that the associated Hamiltonian is positive for particles above transition.

We take the distribution $F(z, p_z; s)$ to be normalized such that integrating over phase space equals unity, and assume that it obeys the Vlasov equation

$$\frac{\partial F}{\partial s} + \frac{\partial \mathcal{H}}{\partial p_z}\frac{\partial F}{\partial z} - \frac{\partial \mathcal{H}}{\partial z}\frac{\partial F}{\partial p_z} = \frac{\partial F}{\partial s} + \{F, \mathcal{H}\} = 0, \tag{2}$$

where the Poisson bracket $\{\cdot, \cdot\}$ is defined as usual and the Hamiltonian

$$\mathcal{H}(z, p_z; s) = \frac{\alpha_c}{2}p_z^2 + V(z; s). \tag{3}$$

Here, α_c is the momentum compaction, while the longitudinal potential $V(z; s)$ includes both the longitudinal focusing provided by rf acceleration and collective effects described by the one-turn wakefield (or impedance) $W_{\|}(z)$ ($Z_{\|}(k)$). Ignoring the damping and diffusion associated with synchrotron emission in Eq. (2) is justified if these effects are slow with respect to both the synchrotron motion and any instability growth rates; Vlasov and Fokker-Planck studies done in Refs.[12,14] have shown that this simplification appears to apply for a wide variety of longitudinal impedances.

To analyze the longitudinal stability of the beam, we write the distribution function as the sum $F(z, p_z; s) = F_0(z, p_z) + F_1(z, p_z; s)$, with F_0 the equilibrium distribution and F_1 the perturbation. In keeping with this linearization, we also separate the longitudinal potential associated with one turn in the ring as $V(z;s) = V_0(z) + V_1(z;s)$. The static potential $V_0(z)$ includes both the rf focusing and any potential-well distortion due to equilibrium wakefields driven by the unperturbed distribution function $F_0(z,p_z)$, while the potential $V_1(z;s)$ contains the wakefields driven by the perturbation $F_1(z,p_z;s)$. Under these assumptions, the single-particle Hamiltonian for longitudinal motion in a storage ring can be written as the sum

$$\mathcal{H}(z,p_z;s) = \mathcal{H}_0(z,p_z) + V_1(z;s), \tag{4}$$

where the equilibrium Hamiltonian $\mathcal{H}_0(z,p_z) \equiv \frac{\alpha_c}{2}p_z^2 + V_0(z)$ is defined such that $\{F_0, \mathcal{H}_0\} = 0$. The perturbing potential V_1 involves the impedance driven by the perturbation F_1, being given by

$$V_1(z;s) \equiv -\frac{2I}{\gamma I_A}\int d\hat{z}d\hat{p}_z\; F_1(\hat{z},\hat{p}_z;s)\int dk\; e^{ik(z-\hat{z})}\frac{Z_\parallel(k)}{ikZ_0}, \tag{5}$$

where I is the average single-bunch current (i.e., the bunch charge times the speed of light divided by the ring circumference), $Z_0 \approx 377\ \Omega$ is the impedance of free space, and $I_A \approx 17$ kA is the Alfvén current.

We begin our analysis by assuming that we have solved the unperturbed problem, by which we mean that we have found the action-angle variables associated with the Hamiltonian $\mathcal{H}_0(z,p_z) = \alpha_c p_z^2/2 + V_0(z)$. This implies that we have computed the canonical transformation $(z,p_z) \to (\Phi,\mathcal{I})$ that results in $\mathcal{H}_0(z,p_z) \to \mathcal{H}_0(\mathcal{I})$ and $F_0(z,p_z) \to F_0(\mathcal{I})$, in which case the linearized Vlasov equation for the perturbation becomes

$$\begin{aligned}\left[\frac{\partial}{\partial s} + \frac{\partial\mathcal{H}_0}{\partial\mathcal{I}}\frac{\partial}{\partial\Phi}\right]F_1(\Phi,\mathcal{I};s) = -\frac{2I}{\gamma I_A}\frac{\partial F_0}{\partial\mathcal{I}}\frac{\partial}{\partial\Phi}\int d\hat{\Phi}d\hat{\mathcal{I}}\; F_1(\hat{\Phi},\hat{\mathcal{I}};s)\\ \times\int dk\; e^{ik[z(\Phi,\mathcal{I})-z(\hat{\Phi},\hat{\mathcal{I}})]}\frac{Z_\parallel(k)}{ikZ_0}.\end{aligned} \tag{6}$$

To make further analytic progress, we will approximate the unperturbed motion as that of a simple harmonic oscillator. In the simplest setting we assume that the unperturbed motion is given entirely by the rf focusing, so that $V_0(z) = \omega_s^2 z^2/2c^2\alpha_c$ with ω_s the synchrotron frequency, but an extension can be made by using the self-consistent bunch length σ_z obtained from the Haïssinski equilibrium solution that includes the wakefields from F_0. For now we assume simple harmonic motion such that $\partial\mathcal{H}_0/\partial\mathcal{I} = \omega_s/c = \alpha_c\sigma_\delta/\sigma_z$, while the position and equilibrium distribution function are given by

$$z = \sigma_z\sqrt{2\mathcal{I}/\langle\mathcal{I}\rangle}\cos\Phi \qquad \text{and} \qquad F_0(\mathcal{I}) = \frac{e^{-\mathcal{I}/\langle\mathcal{I}\rangle}}{2\pi\langle\mathcal{I}\rangle}, \tag{7}$$

respectively, with the mean action $\langle\mathcal{I}\rangle = \sigma_z\sigma_\delta$. Similar harmonic oscillator/Gaussian models have been used by a number of other authors (e.g.,[12,15]) who have shown that this approximation typically results in fairly accurate predictions for single-frequency rf systems even in the presence of potential well distortion.

We return to the linear partial differential equation (6) with the observation that a general solution can be written as a sum of solutions with exponential time dependence, which we isolate by defining $F_1(\Phi, \mathcal{I}; s) = \tilde{F}_1(\Phi, \mathcal{I})e^{-i\Omega s/c}$. Additionally, we introduce the density bunching associated with the perturbation

$$\mathcal{B}(k) = \int d\Phi d\mathcal{I}\ \tilde{F}_1(\Phi, \mathcal{I})e^{-ikz(\Phi,\mathcal{I})}, \tag{8}$$

and then combine these definitions to write the linearized Vlasov equation (6) as

$$\left[-\frac{i\Omega}{c} + \frac{\omega_s}{c}\frac{\partial}{\partial\Phi}\right]\tilde{F}_1(\Phi, \mathcal{I}) = \frac{2I}{\gamma I_A}\frac{e^{-\mathcal{I}/\langle\mathcal{I}\rangle}}{2\pi\langle\mathcal{I}\rangle^2}\int dk\ \frac{Z_\parallel(k)}{ikZ_0}\mathcal{B}(k)\frac{\partial}{\partial\Phi}e^{ikz(\Phi,\mathcal{I})}. \tag{9}$$

This simplifies further if we express z in terms of the action and angle coordinates via (7), and use the Jacobi-Anger identity $e^{ikz(\Phi,\mathcal{I})} = e^{ix\cos\Phi} = \sum_n i^n J_n(ka)e^{in\Phi}$ where $a = \sigma_z\sqrt{2\mathcal{I}/\langle\mathcal{I}\rangle}$. We insert this into Eq. (9), use the fact that $c/\omega_s = \langle\mathcal{I}\rangle/\alpha_c\sigma_\delta^2$, and rewrite the s-derivative to find that

$$\begin{aligned} e^{-i\Omega\Phi/\omega_s}\left[-\frac{i\Omega}{\omega_s} + \frac{\partial}{\partial\Phi}\right]\tilde{F}_1 &= \frac{\partial}{\partial\Phi}\left[e^{-i\Omega\Phi/\omega_s}\tilde{F}_1\right] \\ &= e^{-i\Omega\Phi/\omega_s}\frac{2I}{\alpha_c\sigma_\delta^2\gamma I_A}\frac{e^{-\mathcal{I}/\langle\mathcal{I}\rangle}}{2\pi\langle\mathcal{I}\rangle}\int dk\ \frac{Z_\parallel(k)}{ikZ_0}\mathcal{B}(k) \\ &\qquad\times\sum_{n=-\infty}^{\infty}(in)i^n J_n\left(k\sigma_z\sqrt{2\mathcal{I}/\langle\mathcal{I}\rangle}\right)e^{in\Phi}. \end{aligned} \tag{10}$$

Since $\tilde{F}$ is 2π-periodic in Φ, integrating the left-hand side over Φ from Φ' to $\Phi' + 2\pi$ results in $e^{-i\Omega\Phi'/\omega_s}(e^{-2\pi i\Omega/\omega_s} - 1)\tilde{F}_1(\Phi', \mathcal{I})$, while the right-hand side is similarly straightforward. Cancelling out common terms and setting $\Phi' \to \Phi$, we find that

$$\tilde{F}_1(\Phi, \mathcal{I}) = \frac{Ie^{-\mathcal{I}/\langle\mathcal{I}\rangle}}{\pi\langle\mathcal{I}\rangle\gamma I_A\alpha_c\sigma_\delta^2}\int dk\ \frac{Z_\parallel(k)}{ikZ_0}\mathcal{B}(k)\sum_{n\neq 0}\frac{ni^n e^{in\Phi}}{n - \Omega/\omega_s}J_n\left(k\sigma_z\sqrt{2\mathcal{I}/\langle\mathcal{I}\rangle}\right). \tag{11}$$

We can convert (11) into a closed-form equation for $\mathcal{B}(k)$ by multiplying by e^{-ikz} and integrating over action and angle. Writing the scaled action $\mathcal{I}/\langle\mathcal{I}\rangle = r$ leads to

$$\begin{aligned} \mathcal{B}(k) &= \frac{2I}{\gamma I_A\alpha_c\sigma_\delta^2}\int d\kappa\ \frac{Z_\parallel(\kappa)}{i\kappa Z_0}\mathcal{B}(\kappa)\frac{e^{-r}}{2\pi} \\ &\quad\times\sum_{n,\ell}\frac{ni^{n+\ell}}{n - \Omega\omega_s}\int_0^\infty dr\ e^{-r}J_n\big(\kappa\sigma_z\sqrt{2r}\big)J_{-\ell}\big(k\sigma_z\sqrt{2r}\big)\int_0^{2\pi} d\Phi\ \frac{e^{i(n+\ell)\Phi}}{2\pi} \end{aligned} \tag{12}$$

$$= \frac{2I}{\gamma I_A\alpha_c\sigma_\delta^2}\int d\kappa\ \frac{Z_\parallel(\kappa)}{i\kappa Z_0}\mathcal{B}(\kappa)\int_0^\infty dr\ e^{-r}\sum_{n\neq 0}\frac{nJ_n\big(\kappa\sigma_z\sqrt{2r}\big)J_n\big(k\sigma_z\sqrt{2r}\big)}{n - \Omega/\omega_s}. \tag{13}$$

In addition, the integration over r can be done with the help of Gradshteyn and Ryzhik[16], which lists $\int_0^\infty dx\ e^{-x}J_n(a\sqrt{x})J_n(b\sqrt{x}) = e^{-(a^2+b^2)/4}I_n(ab/2)$, while the sum can be rewritten using

$$\sum_{n\neq 0} \frac{n}{n-y} I_n(x) = \sum_{n=1}^{\infty} \left[I_n(x) + \frac{y}{n-y} I_n(x) - \frac{y}{n+y} I_n(x) \right] \tag{14}$$

$$= e^x - I_0(x) + \sum_{n=1}^{\infty} \frac{2y^2}{n^2 - y^2} I_n(x). \tag{15}$$

Hence, we find that the bunch is governed by the equation

$$\mathcal{B}(k) = \frac{2I}{\gamma I_A \alpha_c \sigma_\delta^2} \int d\kappa \, \frac{Z_\parallel(\kappa)}{i\kappa Z_0} \mathscr{M}(\sigma_z k, \sigma_z \kappa; \Omega) \mathcal{B}(\kappa), \tag{16}$$

where the symmetric kernel $\mathscr{M}$ involves the modified Bessel function $I_n(x)$, being given by

$$\mathscr{M}(x, y; \Omega) = e^{-(x^2+y^2)/2} \left[e^{xy} - I_0(xy) \right] + e^{-(x^2+y^2)/2} \sum_{n\geq 1} \frac{2(\Omega/\omega_s)^2 I_n(xy)}{n^2 - (\Omega/\omega_s)^2}. \tag{17}$$

Although Eqs. (16) and (17) give a compact, relatively simple, and in our opinion mathematically attractive relationship governing the longitudinal stability of a perturbation to wakefields, it is unfortunately not what we can call a solution. To turn this into something we can use, we will take inspiration from Sacherer's analysis[5,6], in which the microwave instability is described as resulting from classical mode coupling. Within this framework the angular modes, which oscillate at harmonics of the synchrotron frequency for $I \to 0$, are frequency-shifted by the impedance as the current is increased. Onset of the microwave instability occurs at the current for which the impedance-induced frequency shift causes two initially real and distinct modes to become degenerate in frequency. At higher currents, the two modes couple into an exponentially damped and an exponentially growing solution whose frequencies are complex conjugates of each other. Indeed, Eq. (16) is consistent with this picture, in that if $\tilde{\mathcal{B}}(k)$ is a solution with frequency $\tilde{\Omega}$, then $\tilde{\mathcal{B}}(-k)^*$ is also a solution with frequency $\tilde{\Omega}^*$, so that the spectrum of (16) contains purely real frequencies and/or ones that come in complex conjugate pairs.

We investigate this further by considering the stability equation (16) for a real frequency $\bar{\Omega}$. In this case Eq. (16) becomes an eigen-problem whose real eigenvalue $\propto 1/I$ gives the current at which the perturbation oscillates $\sim e^{-i\bar{\Omega}z/c}$. By considering a set of $\bar{\Omega}(I)$ we can identify mode coupling with the current at which solutions with real $\bar{\Omega}$ disappear. More explicitly, we discretize the integral in (16) by picking a set of equally spaced wave-vectors, $k_j = -k_{\max} + j\Delta k$ with $j = 0, 1, \ldots, 2k_{\max}/\Delta k$. Then, for a given real $\bar{\Omega}$ Eq. (16) can be expressed as the eigenvalue problem

$$\mathsf{M}_\parallel \mathsf{b} = \lambda \mathsf{b}, \tag{18}$$

where the bunching vector b has components $\mathsf{b}_j = \mathcal{B}(k_j)$, while the matrix $\mathsf{M}_\parallel$ and eigenvalue λ are given by

$$\left(\mathsf{M}_\parallel\right)_{j,\ell} = \Delta k \frac{Z_\parallel(k_\ell)}{ik_\ell Z_0} \frac{2\mathscr{M}\left(\sigma_z k_j, \sigma_z k_\ell; \bar{\Omega}\right)}{\gamma \alpha_c \sigma_\delta^2 I_A}, \qquad \lambda = \frac{1}{I}. \tag{19}$$

Real eigenvalues give physically relevant solutions, and if we take the largest such real $\lambda = 1/I$ over a range of $\bar{\Omega}$ we can identify the microwave instability threshold current as the value of $1/\lambda$ at which two initially distinct solutions merge.

3. Examples using the steady-state CSR impedance

To show how our method works in practice, we will first assume that the impedance is due entirely to steady-state coherent synchrotron radiation (CSR). For a particle travelling in a circle of radius ρ, the steady-state, one-turn CSR impedance is[17]

$$Z_{\parallel}(k) = Z_0 \frac{\Gamma(2/3)}{3^{1/3}} \frac{\sqrt{3} + \mathrm{sgn}(k) i}{2} \left| k\rho \right|^{1/3}, \tag{20}$$

where $\mathrm{sgn}(k) = +1(-1)$ if k if positive(negative), and the Gamma function $\Gamma(2/3) \approx 1.354$. Inserting the CSR impedance (20) into (16) and defining the dimensionless variable $x = k\sigma_z$ results in the following equation for the bunching perturbation:

$$\mathcal{B}(x) = \xi_{\mathrm{CSR}} \frac{\Gamma(2/3)}{3^{1/3}} \int dy \, \frac{\mathrm{sgn}(y)\sqrt{3} + i}{i \left| y \right|^{2/3}} \mathscr{M}(x, y) \mathcal{B}(y). \tag{21}$$

As was discussed in the studies of Refs.[12,14], collective instability to unshielded, steady-state CSR is governed by the single parameter ξ_{CSR}, which we have defined as was done in those references to be given by

$$\xi_{\mathrm{CSR}} \equiv \frac{I(\rho/\sigma_z)^{1/3}}{\gamma I_A \alpha_c \sigma_\delta^2}. \tag{22}$$

The discrete version of (21) can be written as the eigenvalue equation (18) with $\lambda = 1/I$ and the matrix

$$\left(\mathsf{M}_{\parallel}\right)_{j,\ell} = \Delta x \frac{\xi_{\mathrm{CSR}}}{I} \frac{\Gamma(2/3)}{3^{1/3}} \frac{\mathrm{sgn}(x_\ell)\sqrt{3} + i}{i \left| x_\ell \right|^{2/3}} \mathscr{M}(x_j, x_\ell; \bar{\Omega}). \tag{23}$$

Having written out our formalism for the CSR impedance, we now proceed to show how to obtain a prediction for the microwave instability threshold. First, we observe that at harmonics of the synchrotron frequency, $\bar{\Omega} = n\omega_s$ for natural numbers n, all the matrix elements associated with $\mathscr{M}$ from Eq. (17) diverge and $\lambda = 1/I \to \infty$. These solutions correspond to the zero-current modes that oscillate at $n\omega_s$. If we choose $\bar{\Omega}$ to depart from a synchrotron harmonic, for example, $\bar{\Omega} = \omega_s(1 + \epsilon)$ with $0 < \epsilon < 1$, the spectrum of $\mathsf{M}_{\parallel}$ has both real λ's and ones that come in complex conjugate pairs. Since $\lambda = 1/I$ only real eigenvalues have physical meaning, and the largest such λ gives the smallest current capable of supporting stable oscillations with real frequency $\bar{\Omega}$. We plot these values of $\xi_{\mathrm{CSR}} \propto 1/\lambda$ as a function of $\bar{\Omega}/\omega_s$ in Fig. 1. The Figure shows that as the current $I \propto \xi_{\mathrm{CSR}}$ is increased, the mode frequencies are shifted by the impedance from harmonics of ω_s, so that eventually the mode with initial frequency $n\omega_s$ collides with its neighbors at $(n \pm 1)\omega_s$. This mode coupling occurs between each harmonic of ω_s, and we identify the instability threshold $\xi_{\mathrm{CSR}}^{\mathrm{thresh}}$ with the minimum value of ξ_{CSR} for which

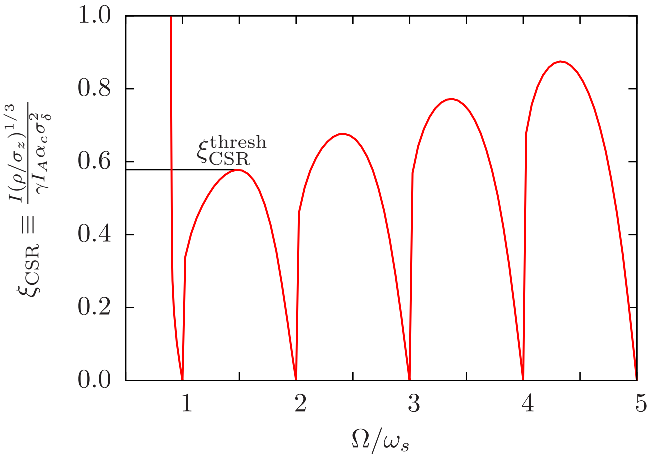

Fig. 1. Example of longitudinal mode-coupling induced by the steady state CSR impedance (20). The natural modes oscillate at harmonics of the synchrotron frequency at zero current, but approach and merge as the current increases. The first such mode coupling occurs between the fundamental and first harmonic when the scaled current $\xi_{\mathrm{CSR}}^{\mathrm{thresh}} \approx 0.578$, which we identify with the microwave instability threshold.

two modes merge. As shown in Fig. 1, the onset of the microwave instability is predicted to occur when the initial $\Omega = \omega_s$ mode couples to one with $\Omega = 2\omega_s$ at a value of $\xi_{\mathrm{CSR}}^{\mathrm{thresh}} \approx 0.578$. The theshold agrees fairly well with that obtained by Fokker-Planck simulations, which show that $\xi_{\mathrm{CSR}}^{\mathrm{thresh}} \approx 0.5$ [12,14].

In addition, the mode-coupling diagram of Fig. 1 looks very similar to that plotted in Ref. [12], with precisely the same predicted value of $\xi_{\mathrm{CSR}}^{\mathrm{thresh}} \approx 0.578$. This is perhaps not surprising, since for this example Ref. [12] also employed a harmonic oscillator/Gaussian approximation to the longitudinal motion. Their solution was obtained by expanding the perturbation $\tilde{F}_1(\Phi, \mathcal{I})$ as a sum of orthogonal polynomials, truncating the sum, and then solving the resulting eigenvalue problem for the mode shapes and frequencies.

Our previous results assumed that the CSR is produced in vacuum, but in a real accelerator the vacuum chamber walls effectively limit the radiation produced at low frequencies. If we assume the next simplest model, in which the vacuum chamber can be approximated as two parallel planes separated by a distance $2h$, then the shielded CSR impedance becomes a function of both ξ_{CSR} and the shielding parameter $\Pi = \sigma_z \rho^{1/2}/h^{3/2}$, being given by

$$\begin{aligned} Z_\parallel(y) = Z_0 \frac{\Gamma(2/3)}{3^{1/3}} \frac{\sqrt{3} + \mathrm{sgn}(y) i}{2} \frac{|y|^{1/3}}{(\sigma_z/\rho)^{1/3}} \\ \times \frac{\sqrt{\pi}[\sqrt{3} + \mathrm{sgn}(y) i]\Pi^{2/3}}{\Gamma(2/3)\,|y|^{2/3}} \sum_{p=0}^{\infty} \int_0^\infty dt\, e^{-t^3} \exp\left[-t \tfrac{(2p+1)^2 3^{1/3} \pi^2 \Pi^{4/3}}{2(1-i\sqrt{3})|y|^{2/3}}\right]. \end{aligned} \tag{24}$$

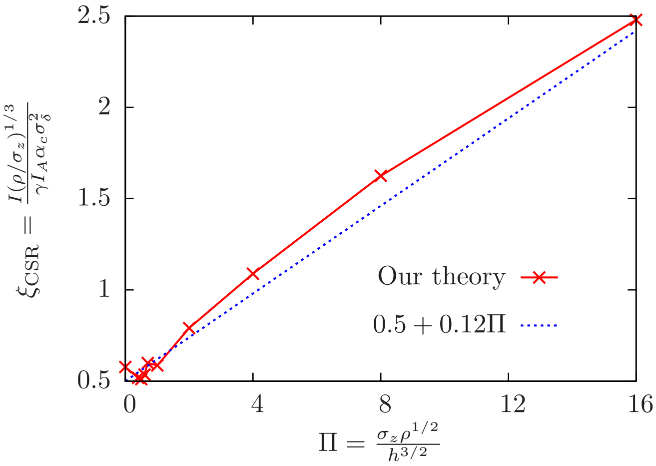

Fig. 2. Prediction of the microwave instability threshold as a function of the shielding parameter. The dotted line is the fit for $\Pi \gtrsim 1$ given in Ref.[14].

Note that the second line quantifies the shielding; it significantly reduces the unshielded CSR impedance for frequencies $|k| \lesssim \Pi/\sigma_z$, while asymptoting to unity for $|k| \gg \Pi/\sigma_z$.

Using our model to investigate stability to the steady-state shielded CSR impedance (24) proceeds much like what we presented for the unshielded case, although the calculations become more numerically intense larger. In particular, since the predicted mode coupling occurs at a frequency that scales linearly with Π, calculating the instability threshold becomes ever more computationally intensive for larger shielding parameters. We plot the results of our theory in Fig. 2. We see that the theory largely follows the straight line fit (dotted line) found in Ref.[14], so that the theory also applies when shielding is strong. On the other hand, the theory shows no evidence of the deviation from the straight scaling that has been observed in simulations[14] and experiments[18] near $\Pi \approx 0.6$. In particular, those previous publications found that the threshold drops significantly below the line for $0.4 \lesssim \Pi \lesssim 0.8$, and that over this range the instability is weak and depends in detail on the longitudinal damping and diffusion rates. Since our theory neglects the details of damping, we do not expect it to capture these physics.

4. Conclusions

We derived an integral equation governing longitudinal stability, and shown how to apply the physics of mode-coupling to find the microwave stability threshold current. Applications of this theory to both shielded and unshielded CSR impedances display good agreement over a wide range of parameters, provided the instability is not weak. Future work will apply the theory to other impedances including that of a broad-band resonator and the impedance model of the Advanced Photon Source.

Acknowledgments

I would like to thank our colleagues at the APS (particularly M. Borland, L. Emery, V. Sajaev, M. Sangroula, and A. Zholents), and other at the workshop (including G. Bassi, A. Blednykh, P. Kruske, and R. Warnock) for useful discussions and important suggestions. This work was supported by U.S. Dept. of Energy Office of Sciences under Contract No. DE-AC02-06CH11357.

References

1. A. W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators* (Wiley, New York, 1993).
2. D. Boussard, *Observation of microwave longitudinal instabilities in the CPS*, Report LabII/RF/INT/75-2, CERN (1975).
3. V. K. Neil and A. M. Sessler, Longitudinal resistive instabilities of intense coasting beams in particle accelerators, *Rev. Sci. Instrum.* **36**, p. 429 (1965).
4. E. Keil and W. Schnell, *Concerning longitudinal stability in the ISR*, Report ISR/TH/RF/69-48, CERN/SPS (1969).
5. F. Sacherer, A longitudinal stability criterion for bunched beams, *IEEE Trans. Nucl. Sci.* **20**, p. 825 (1973).
6. F. Sacherer, Bunch lengthening and microwave instability, *IEEE Trans. Nucl. Sci.* **24**, p. 1393 (1977).
7. J. M. Wang and C. Pelligrini, On the condition for a single bunch high frequency fast blow-up, in *Proc. XI Int. Conf. on High Energy Accel.*, (Geneva, Switzerland, 1980). BNL-28O34.
8. S. Krinsky and J. M. Wang, Longitudinal instabilities of bunched beams subject to a non-harmonic rf potential, *Particle Accel.* **17**, p. 108 (1985).
9. T. Suzuki, Y. Chin and K. Satoh, Mode coupling theory and bunch lengthening in SPEAR II, *Particle Accel.* **13**, p. 179 (1983).
10. J. L. Laclare, High current single bunch transverse instabilities at the ESRF: a new approach, in *CERN accelerator school: advanced accelerator physics*, (Oxford, England, 1985).
11. K. Oide and K. Yokoya, *Longitudinal Single-Bunch Instability in Electron Storage Rings*, Report 90-10, KEK (1990).
12. Y. Cai, Linear theory of microwave instability in electron storage rings, *Phys. Rev ST Accel. Beams* **14**, p. 061002 (2011).
13. R. D. Ruth and J. M. Wang, Vertical fast blow-up in a single bunch, *IEEE Trans. Nucl. Sci.* **28**, p. 2405 (1981).
14. K. L. F. Bane, Y. Cai and G. Stupakov, Threshold studies of the microwave instability in electron storage rings, *Phys. Rev ST Accel. Beams* **13**, p. 104402 (2010).
15. S. Petracca, T. Demma and K. Hirata, Gaussian approximation of the bunch lengthening in electron storage rings with a typical wake function, *Phys. Rev ST-Accel. Beams* **8**, p. 074401 (2005).
16. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th edn. (Academic Press, 2000).
17. A. Faltens and L. J. Laslett, *An Estimate of Limits to the Longitudinal Coupling Impedance*, Report 20550, BNL (1975).
18. P. Kuske, CSR-driven longitudinal single bunch instability thresholds, in *Proc. of IPAC*, (Shangai, China, 2013).